Bessel Functions

Bessel's equation:

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

Where we have used the notation: $y(x)' = dy/dx$, $y(x)'' = d^2y/dx^2$, etc.

Dividing by x^2:

$$y'' + P(x)y' + Q(x)y = 0$$

Where $P(x) = 1/x$ and $Q(x) = (x^2-p^2)/x^2$.

Therefore: $xP(x) = 1$ and $x^2Q(x) = -p^2 + x^2$ and the origin is a singular point (i.e. these functions are not well behaved at $x = 0$ since they can not be defined at $x = 0$). This singular point is a regular singular point, regular since $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$ (i.e. they can be represented by a convergent series at $x = 0$).

An equation of the general form $y'' + P(x)y' + Q(x)y = 0$ with a regular singular point at $x = 0$ can be solved using the Frobenius method of power series substitution. A Frobenius series is one of the form:

$$y = x^m(a_0 + a_1x + a_2x^2 + \dots) = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots,$$

which has an infinite number of terms and m is a number that can be found from the indicial equation:

$$m(m - 1) + mp_0 + q_0 = 0,$$

where p_0 and q_0 are obtained from the power series expansions:

$$xP(x) = \sum_{n=0}^{\infty} p_nx^n, \text{ and } x^2Q(x) = \sum_{n=0}^{\infty} q_nx^n, \quad \text{for } n = \{0, 1, 2, \dots\}.$$

Since $xP(x) = 1$, $p_0 = 1$ and since $x^2Q(x) = -p^2 + x^2$, $q_0 = -p^2$, therefore the indicial equation is:

$$m^2 - p^2 = 0.$$

The roots of this equation, called the exponents, are m1 = p and m2 = -p. Therefore, there is a solution of the form:

$y = x^p\sum a_nx^n = \sum a_nx^n = p$, $a_0 \neq 0$ and the power series, $\sum a_nx^n$ converges for all x.

Using this power series expansion for y, the first differential of y becomes:

$$y' = \sum(n+p)a_nx^{n+p-1},$$

and the second differivative becomes:

$$y'' = \sum (n+p-1)(n+p)a_nx^{n+p-2}.$$

This gives us the following expressions for Bessel's equation:

$$x^2y'' = \Sigma\ (n+p-1)(n+p)a_nx^{n+p},$$

$$xy' = \Sigma\ (n+p)a_nx^{n+p},$$

$$x^2y = \Sigma a_{n-2}x^{n+p},$$

$$-p^2y = \Sigma -p^2a_nx^{n+p}.$$

Adding the above series: $x^2y'' + xy' + (x^2 - p^2)y = 0$, and equating to zero the coefficients of x^{n+p} for each n:

$$n: (n+p-1)(n+p)a_n + (n+p)a_n + a_{n-2} - p^2a_n = 0.$$

$$n=0: (p-1)pa_0 + pa_0 + a_{-2} - p^2a_0 = 0, \quad a_{-2} = 0.$$

$$n=1: p(1+p)a_1 + (1+p)a_1 + a_{-1} - p^2a_1 = 0,$$

$$\Rightarrow pa_1 + p^2a_1 + a_1 + pa_1 + a_{-1} - p^2a_1 = 0,$$

$$\Rightarrow (2p+1)a_1 = -a_{-1} = 0,$$

$$\Rightarrow a_1 = -a_{-1}/(2p+1) = 0.$$

$$n=2: (1+p)(2+p)a_2 + (2+p)a_2 + a_0 - p^2a_2 = 0,$$

$$\Rightarrow 2a_2 + 3pa_2 + p2a_2 + 2a_2 + pa_2 - p2a_2 = -a_0,$$

$$\Rightarrow a_2(4+4p) = -a_0,$$

$$\Rightarrow a_2 = -a_0/(n(2p+n)).$$

The following general pattern formula emerges for the coefficients:

$$a_n = \frac{-a_{n-2}}{n(2p+n)}$$

$a_0 \neq 0$ but is otherwise arbitrary, $a_{-1} = 0$, $a_1 = 0$, all the odd terms equal zero.
Applying the formula for a_n:

$$a_0, \quad a_2 = \frac{-a_0}{2(2p+2)}, \quad a_4 = \frac{-a_2}{4(2p+4)} = \frac{a_0}{2.4(2p+2)(2p+4)},$$

$$a_6 = \frac{-a_4}{6(2p+6)} = \frac{a_0}{2.4.6(2p+2)(2p+4)(2p+6)}, \ \dots$$

<u>Solution</u>:

$$y = a_o x^p(1 \;-\; x^2/[2^2(p+1)] \;+\; x^4/[2^4 2!(p+1)(p+2)] \;-\; x^6/[2^6 3!(p+1)(p+2)(p+3)] \;+\; \ldots)$$

$$= a_o x^p \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n!(p+1)\ldots(p+n)} \;.$$

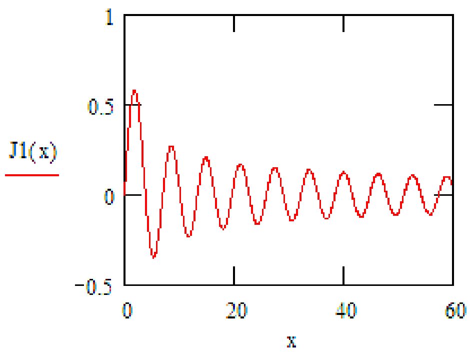
The Bessel function of the first kind of prder p, $J_p(x)$ is found by putting $a_0 = (1/2)^p p!$:

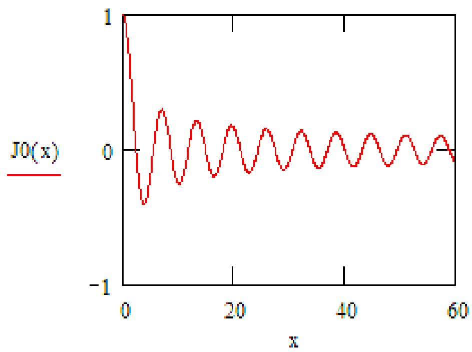
$$J_p(x) \;=\; \frac{x^p}{2^p p!} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n!(p+1)\ldots(p+n)}$$

Which simplifies to:

$$J_p(x) \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+p}}{n!(p+n)!}$$

This function is plotted below for p = 1 (left) and p = 0 (right):



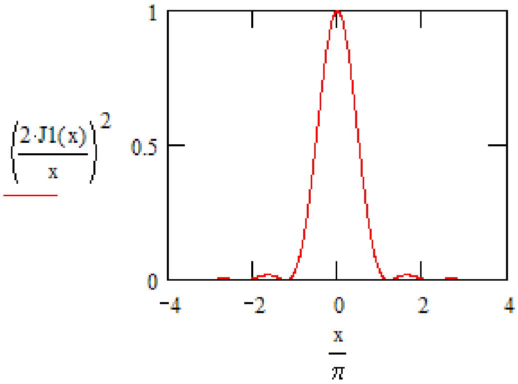


These functions have the appearance of damped sine and cosine functions. Bessel functions have numerous applications in physics and maths. Many applications involving damped waves or oscillations make use of the Bessel functions. For example, consider $J_0(x)$, the intensity of light that undergoes Fraunhofer diffraction through a narrow circular aperture, as seen on a screen, is given by:

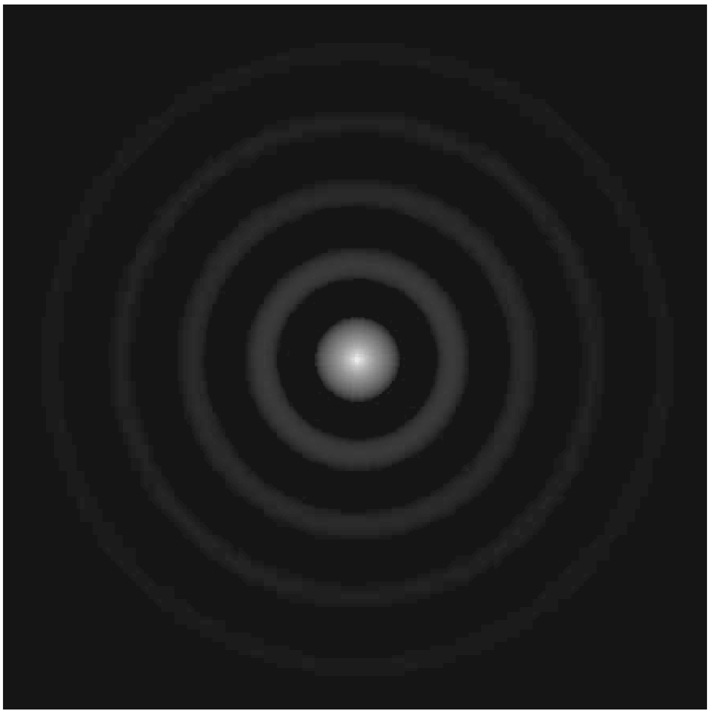
$$I = I_0 \left[\frac{2J_1(x)}{x} \right]^2$$

where I_0 is the intensity of the incident (non-diffracted) light beam.

This intensity function is plotted below:



Plotting this function in 2D (using plane polar coordinates) gives us the pattern of light diffracted through a small circular aperture and projected onto a screen:



Notice that what we see is the central beam of light at incident intensity (I_0) in the centre of the image and a series of concentric diffraction rings fading in intensity away from the central beam. This function was plotted in greyscale (left) and inverse greyscale (right) using a Windows application written in VC#. It uses the first 40 or so terms of the Bessel function series. However, even with 40 (or 100) terms the solution diverges at large values of x (that is for large radial distance, r, from the centre) but gives us four rings. More exact algorithms are available (e.g. the VC# extended maths calss available at: http://www.codeproject.com/KB/cs/SpecialFunction.aspx).

This more exact algorithm will give up to 6 rings before the rings fade from view in greyscale.