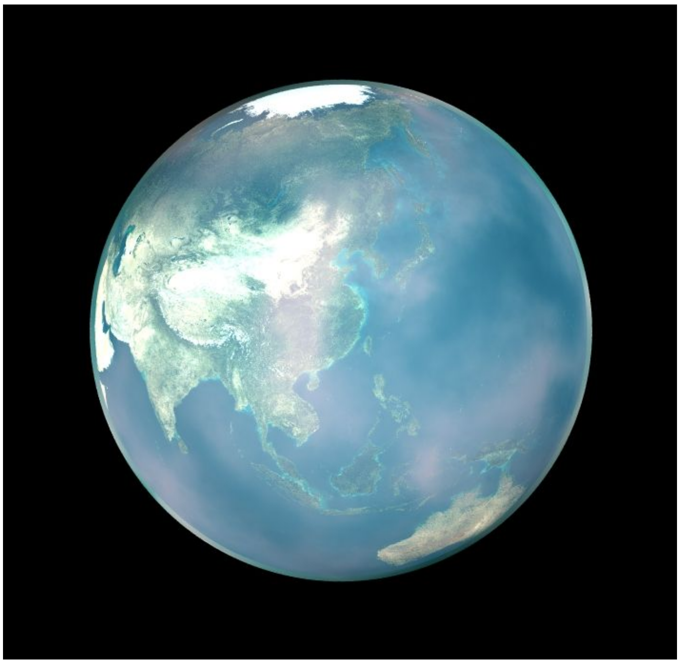
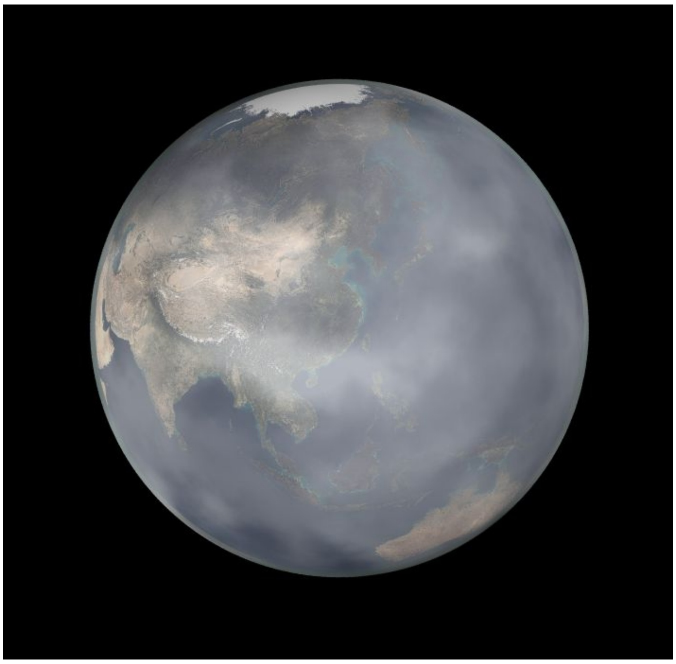
Special relativity

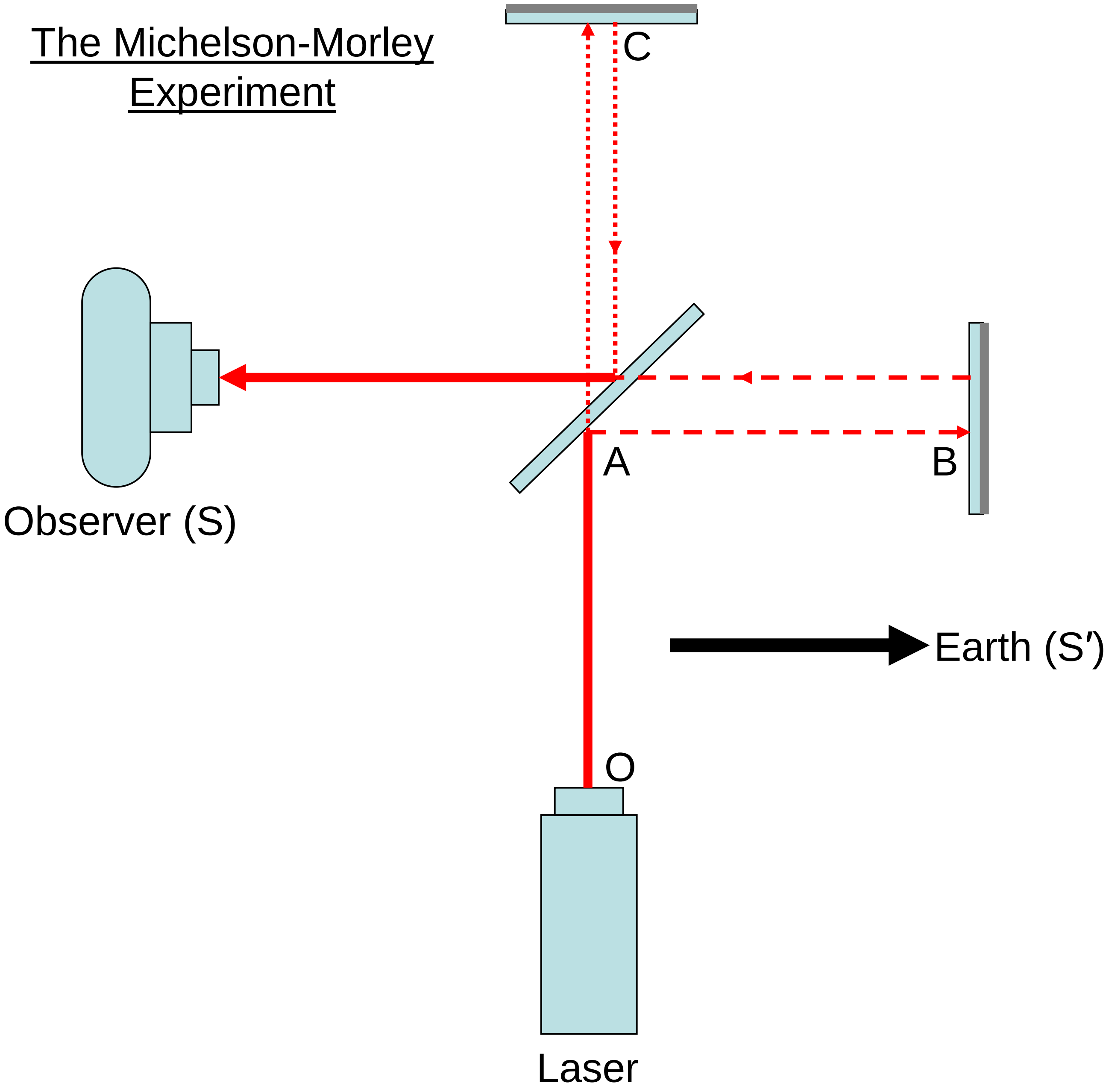
According to the Twin Paradox, a space traveler leaving his twin brother behind on Earth, might return some years later to find that his twin has aged much more than he has, or, if he spent a lifetime in space, he may perhaps find that humanity has evolved into something quite different, or perhaps that the Earth is barren and dead! Study this article and find out why!

In this article we explore some aspects of Einstein's Theory of Special relativity. This article includes quite a bit of algebra, but those who do not want to study the maths can still learn a lot by skipping the algebra and simply reading the text.

The Michelson-Morley experiment

Michelson and Morley conducted a classic experiment to determine the speed of motion relative to what was then considered to be the ether or Cosmic background substance. In other words, they assumed that the Universe has some fixed velocity relative to which the Earth was moving. To determine this motion they constructed an **interferometer** using laser light. When two light beams that are out of phase are added together (superposed) then an interference pattern results. Two light beams that are originally in phase may become out of phase if one of them travels further by a distance that is not a whole number multiple of wavelengths, with the difference in path length divided by wavelength leaving a remainder. Analysis of the interference pattern, therefore, tells us how much further one beam has travelled than the other.

The Michelson-Morley Experiment



Consider the following two light paths each from L (laser):

•Light path P1: ABA, from A (half-silvered mirror) to B (mirror) to A (half-silvered mirror)

•Light path P2: ACA, from A to C to A,

and on to S (camera/observer). (Note that distances LA, L to A, and AS, A to S, are the same and so we can ignore them when comparing the two light paths).

Note: distance OA = distance AB, call this distance L.

Case 1. Due to the motion of the Earth, the speed of light relative to the apparatus is (naively) expected to change, as is the actual path-length traveled by the light beams.

Case 2. If we then rotated the apparatus through 90 degrees, we would again expect a difference, but in the opposite sense to case (1). Adding the results together should double the difference, making it quite obvious!

Result: No difference in path length was observed, P1 = P2! This implies that the beams maintain a constant speed in both directions! The velocity of the Earth has made no difference!

We expect :

The time taken for the beam to travel AB,

$$t_{AB} = \frac{L}{v_{rel}} = \frac{L}{c-v}$$

The length traveled by the beam from A to B,

$$l_{AB} = ct_{AB} = \frac{Lc}{c-v}$$

Where :

v_{rel} = velocity of beam relative to apparatus moving with the Earth,

c = speed of light (approximately the speed of light in air is the same as that in a vacuum),

v = speed of apparatus (the velocity of the Earth's surface).

The time and length of the return leg from B to A is expected to be :

$$t_{BA} = \frac{L}{c+v},$$

$$l_{BA} = ct_{BA} = \frac{Lc}{c+v},$$

and the total path length traveled by the light from A to B to A :

$$l_{ABA} = \frac{Lc}{c\text{-}v} + \frac{Lc}{c+v},$$

$$= \frac{Lc(c+v)}{(c\text{-}v)(c+v)} + \frac{Lc(c-v)}{(c+v)(c-v)},$$

$$= \frac{Lc^2 + Lcv}{c^2 - v^2} + \frac{Lc^2 - Lcv}{c^2 - v^2} = \frac{2Lc^2}{c^2 - v^2},$$

$$\text{so} : l_{ABA} = \frac{2Lc^2}{c^2 - v^2}$$

Defining :

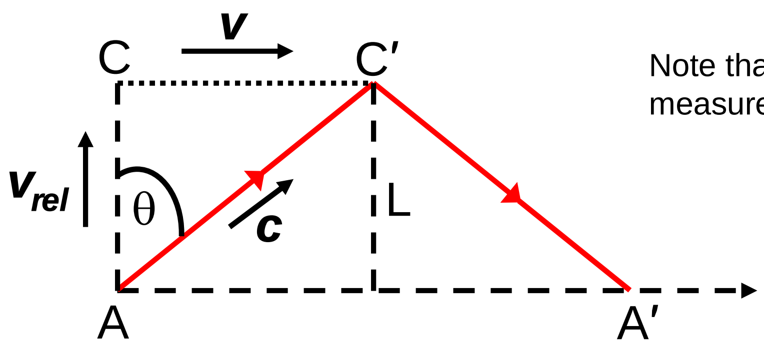
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}},$$

So :

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}} = \frac{c^2}{c^2 - v^2},$$

$$\boxed{l_{ABA} = 2L\gamma^2}$$

For the beam traveling along P2 (ACA), the motion of the Earth increases the path length which becomes diagonal relative to the Earth's surface, as by the time the light reaches the mirror at C, this mirror has moved along slightly to the right (to position C′) due to the motion of the Earth's surface. By the time the light beam returns to A, A has moved twice as far, to A′ as shown below:



Note that length AC = length L, as measured in the lab.

The relative velocity of the beam along AC is:

$$\boldsymbol{v_{rel}} = \boldsymbol{c}\cos\theta, \quad \text{so}: \cos\theta = \frac{\boldsymbol{v_{rel}}}{\boldsymbol{c}}$$

$$\sin\theta = \frac{\boldsymbol{CC'}}{\boldsymbol{AC}} = \frac{\boldsymbol{v}}{\boldsymbol{c}},$$

$$(\text{using}: \sin^2\theta + \cos^2\theta = 1):$$

$$\boldsymbol{v_{rel}} = \boldsymbol{c}\cos\theta = \boldsymbol{c}\sqrt{1 - \sin^2\theta}$$

$$= \boldsymbol{c}\sqrt{1 - \frac{\boldsymbol{v}^2}{\boldsymbol{c}^2}} = \frac{\boldsymbol{c}}{\gamma}$$

The time taking for the beam to travel from A to C is expected to be:

$$\mathrm{t_{AC}} = \frac{\boldsymbol{L}}{\boldsymbol{v_{rel}}},$$

and (also using: velocity = distance/time, and $\boldsymbol{v_{rel}} = \frac{\boldsymbol{c}}{\gamma}$):

$$\boldsymbol{l_{AC}} = \boldsymbol{ct_{AC}} = \frac{\boldsymbol{Lc}}{\boldsymbol{v_{rel}}} = \boldsymbol{L}\gamma$$

and, as $\boldsymbol{l_{AC}} = \boldsymbol{l_{CA}}$:

$$\boxed{\boldsymbol{l_{ACA}} = 2\boldsymbol{l_{AC}} = 2\boldsymbol{L}\gamma}$$

Now comparing the two path lengths we see that they are not equal :

$$\boldsymbol{l}_{ABA} - \boldsymbol{l}_{ACA} = 2\boldsymbol{L}\gamma^2 - 2\boldsymbol{L}\gamma = 2\boldsymbol{L}(\gamma^2 - \gamma)$$

$$= 2\boldsymbol{L}\left(\frac{\boldsymbol{v}^2}{2\boldsymbol{c}^2}\right) = \frac{\boldsymbol{L}\boldsymbol{v}^2}{\boldsymbol{c}^2},$$

where we have used :

$$\gamma = 1 + \frac{\boldsymbol{v}^2}{2\boldsymbol{c}^2}, \quad \text{and} \quad \gamma^2 = 1 + \frac{\boldsymbol{v}^2}{\boldsymbol{c}^2},$$

$$\gamma^2 - \gamma = \left(1 + \frac{\boldsymbol{v}^2}{\boldsymbol{c}^2}\right) - \left(1 + \frac{\boldsymbol{v}^2}{2\boldsymbol{c}^2}\right) = \frac{\boldsymbol{v}^2}{\boldsymbol{c}^2} - \frac{\boldsymbol{v}^2}{2\boldsymbol{c}^2} = \frac{\boldsymbol{v}^2}{2\boldsymbol{c}^2}.$$

To summarise, the expected pathlength difference traveled by the light beams is :

$$\boxed{\frac{\boldsymbol{L}\boldsymbol{v}^2}{\boldsymbol{c}^2}}$$

What Michelson and Morley actually found was that there was **no path difference** at all! That is:

$$\boxed{\boldsymbol{l}_{ABA} = \boldsymbol{l}_{ACA}}$$

So, what went wrong? Nothing!

The Principle of Relativity states:

The laws of physics (including the behaviour of light) must be exactly the same for any two observers moving with constant velocity relative to each other.

Thus, if the velocity of light (in a vacuum) is constant to one observer, then it must be constant to another observer somewhere else. We were wrong in our calculations of the relative velocity of the light beams relative to the apparatus – it can not change!

As we will see later, there are other important reasons why the speed of light (in a vacuum) is constant.

Note: the speed of light in different media is different, but under identical conditions the speed of light is constant and in a vacuum it is always = c, which is approximately 2.998×10^8 m/s.

Observers and Inertial Reference Frames – it's all relative!

An important concept to understanding relativity is the concept of **reference frames**. If I was standing still in a park and you were whizzing around on a merry-go-round, then we would have very different frames of reference. I see you revolving against the still background of the park. You see the world spinning around you! Of course, we would not expect the laws of physics to be any different on the merry-go-round as they are anywhere else – apart from the fact that the merry-go-round is spinning there is nothing special about. The laws of angular momentum and **centrifugal force** apply equally anywhere in the park.

Relativity states that different observers must see the same laws of physics. However, acceleration does cause differences! An object accelerates if it alters its speed and/or course – the merry-go-round is accelerating because it spins, even if there was no friction and it rotated at a fixed speed indefinitely. (Of course in reality it will lose energy and momentum through friction with the air and the axis and so gradually slow down). Rotation can alter the apparent force of gravity. The centrifugal force of the merry-go-round acts a bit like gravity except it is pulling you sideways – if the merry-go-round spins too fast hold on tight, otherwise the centrifugal force might send you flying across the playground!

Special relativity is special because it simplifies things by assuming that neither observer is accelerating. We say that each observer is an **inertial reference frame**. An inertial reference frame is one that is not accelerating, though it can move at constant velocity. For example, as I stand still in the park, I am in an inertial reference frame (ignoring the fact that the Earth is rotating and orbiting and so accelerating) as is somebody who passes by on a train, so long as the train is not accelerating.

General relativity incorporates the effects of acceleration and has a lot to do with gravity. This is a much more mathematically complex theory and will be discussed in another article.

Now, back to our experiment. We expected path P1 (ABA) to be longer than path P2 (ACA). That is we expected the path in the direction of the motion (ABA) to be longer than the path perpendicular (at right angles to) the motion. (Remember, our apparatus was moving to the right in this case.)

The Lorentz Contraction – Moving Rulers Shorten!

What has evidently happened is that the length along the path ABA has shortened, such that the two paths are the same length.

If we define the length of any path parallel to the direction of motion as $L_{\parallel}$ and any path perpendicular to the motion as $L_{\perp}$ then are expected path-lengths were:

$$\mathbf{l}_{ABA} = 2L_{\parallel}\gamma^2$$

$$\mathbf{l}_{ACA} = 2L_{\perp}\gamma$$

However, we actually observed:

$\mathbf{l}_{ABA} = \mathbf{l}_{ACA}$

that is:

$L_{\perp} = L_{\parallel}\gamma$

The path parallel to motion contracts according to:

$L_{\parallel} = L_{\parallel}/\gamma$

This is the Lorentz contraction:

Lorentz contraction :

$$\boldsymbol{L} = \frac{\boldsymbol{L}}{\gamma}$$

So the path-length in the direction of motion, as seen by the light-beam, has been contracted. Moving objects contract along their direction of motion, and the faster they move, the more they contract! Moving objects shrink! This is the Lorentz contraction, and γ is the Lorentz factor.

However, things are more subtle than this. By the Principle of Relativity, if an observer in the lab sees the path followed by the light beam shorten, then an observer moving with the light beam sees the lab (in the direction that the lab appears to be moving past the observer) shorten! No point of reference is special, the laws of physics appear the same to each observer and as neither observer can be sure that they are truly stationary (relative to what?) all motion must be relative.

The Lorentz contraction can also be written as :

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}$$

where

L is the contracted length as measured by our 'stationary' observer;

L_0 is the length as measured by our moving observer,

who does not notice the contraction.

Time Dilation – Moving Clocks Run Slow!

What if we measured time instead of length? There is no denying that the perpendicular path ACA became lengthened due to the earth's motion (the beam followed diagonal paths to reach the mirror) and as the speed of light is constant we would expect the light to take longer to traverse this path than if the earth was stationary (relative to the light beam). (The actual distance between A and C does not alter). However, from the point of view of the light beam it is travelling in a straight line, since it is moving with the apparatus which to an observer on the light beam would appear stationary. This would amount to an apparent slowing down of the light-beam as measured by an observer moving with it and this violates the Principle of Relativity!

The solution to this paradox is time dilation. Moving clocks run slow, so time from the point of view of the light beam has slowed. This means that when one second of time, as measured by someone riding with the beam, elapses, the elapsed time interval according to our 'stationary' observer in the lab is much greater. Time dilation ensures that the Principle of relativity is upheld – the time slows by the exact amount to conceal any change in path length (due to the Earth's motion and not Lorentz contraction for path ACA) so that the speed of light remains unchanged.

Time dilation is relative. If two space ships were drifting toward one another through the void of space and unable to measure their speeds, then they could not tell if they were both moving toward one-another or if one was stationary as the other flies past it. All they could measure is their relative velocity. If the crew on one spaceship think they are stationary, then they will conclude that the other spaceship is moving, and vice versa. What Michelson and Morley's experiment proved was that there was no way to measure the absolute velocity for the Earth relative to the background of space – the Lorentz contraction and time dilation 'prevent its measurement! We do not know what stationary means. All we can do is measure the earth's velocity relative to the Sun or to other stars, all of which are moving relative to one-another! No reference frame is privileged by being at rest and velocity is only relative! Like Lorentz contraction, it depends on perspective. If each ship considers itself at rest, then it will appear as if time on the other ship (which is 'moving') has slowed. Each crew observe the same phenomena happening to the other space ship. It is the relative motion of the ships, rather than the motion of any one in particular, which will cause Lorentz contraction and time dilation to be observed to the other 'moving' spaceship. (We will see that General Relativity introduces a subtle difference by taking acceleration into account – see The Twin Paradox section).

The time dilation is given by:

Time dilation:

$$\boldsymbol{t}' = \frac{\boldsymbol{t}}{\gamma}$$

where:

$\boldsymbol{t}'$ is the time as measured by an observer (S′) in a reference frame moving at constant velocity relative to the 'rest frame';

$\boldsymbol{t}$ is the time as measured by an observer (S) in the rest frame.

Of course we can not say that a frame is really at rest,

rather we are talking in relative terms!

The time dilation may also be written as:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{\boldsymbol{v}^2}{\boldsymbol{c}^2}}}$$

where

$\Delta\boldsymbol{t}$ = the dilated time, t, as observed by our 'stationary' observer;

$\Delta t^0 = t'$, the time interval as observed by our moving observer who does not notice the dilation.

This can seem confusing, as it almost appears to contradict the first time dialtion equation,

but it does not - one just needs to understand who is observing what!

The Twin Paradox

Since time dilation, like Lorentz contraction, depends on the relativistic (Lorentz) factor γ (gamma), it becomes more important at higher velocities. At the kinds of everyday velocities we deal with (less than ~100 m/s in most cases and up to about 500 m/s in a fast passenger aircraft like Concorde, which is only about 0.0002% of the speed of light!) the Lorentz contraction and time dilation are not noticeable. (Although an atomic clock on board Concorde was shown to lose a fraction of a second in accordance with time dilation). However, for a spacecraft travelling at a significant fraction of the speed of light both effects become significant.

An example of the twin paradox is calculated and plotted below. In this example one twin goes on a space voyage and travels at 99.99% the speed of light. The graph shows the time passed in space, according to the traveler (vertical axis) against time elapsed according to the twin left on Earth (horizontal axis). As you can see, the space-traveling twin spends ten years of their time in space but when they return to Earth they find that more than 200 years have elapsed on Earth! The twin who went into space has aged much less! (In effect the twin in space has travelled forward in time).

This might seem odd, since to the twin in the spaceship it is the one on earth who is travelling and so time should slow more for the twin on Earth – right? Wrong! Acceleration imparts an asymmetry: the twin in the spaceship uses energy (fuel) to accelerate and decelerate during their voyage. Special relativity assumes that both twins remain at constant velocity and that neither accelerates. The twin paradox is explained fully by General Relativity which accounts for the effects of acceleration 9and energy expenditure).

A slightly more involved calculation shows that if our space traveler accelerated at a constant acceleration equal to Earth's gravity (g), assuming limitless fuel, then within a years their speed would be very close indeed to the speed of light (though it can never quite reach it). In this case:

•***A 52 year-round trip for our astronaut would see 1,300,000 years elapse on Earth!***

The Twin Paradox

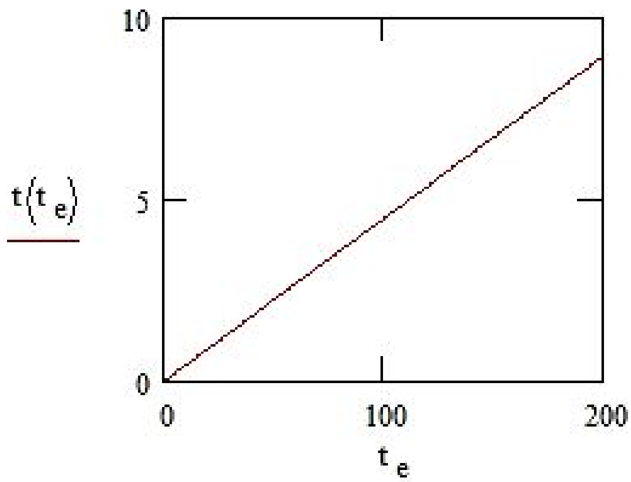
$t' := \frac{t}{\gamma(v)}$ (time dilation equation)

$t_e := 1, 2 .. 1000$ (length of voyage in years by Earth time)

$w := \gamma(0.999) \rightarrow 22.366272042129222171$ (Lorentz factor)

$t(t_e) := \frac{t_e}{w}$ (time dilation)

Time elapsed in space versus time elapsed on Earth



(time in years)

The Lorentz or relativistic factor (gamma, γ) is so important that it is useful to look at its properties, let us see how it's value varies with velocity, v.

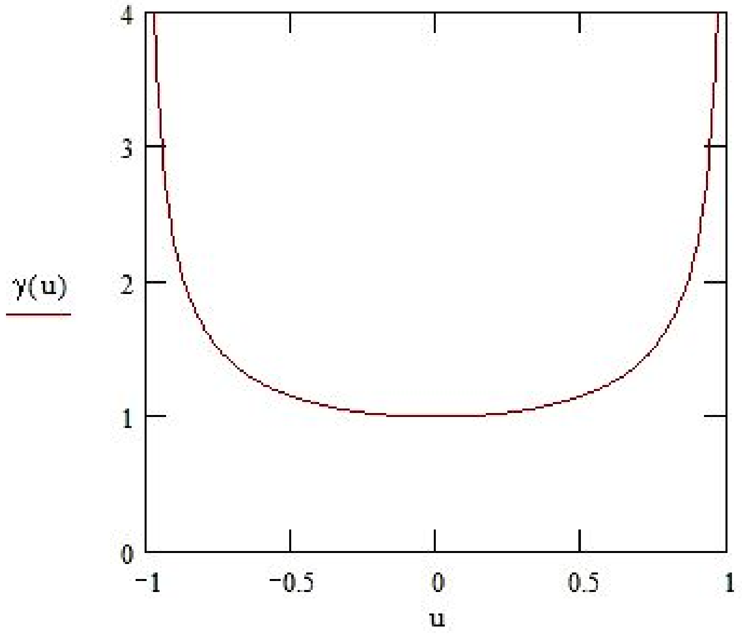
The Relativistic factor

$c := 1$ speed of light = 2.998*10^8 m/s

$u := -1, -0.99 .. 1$ velocity of spacecraft relative to c

$$\gamma(v) := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$ relativistic factor

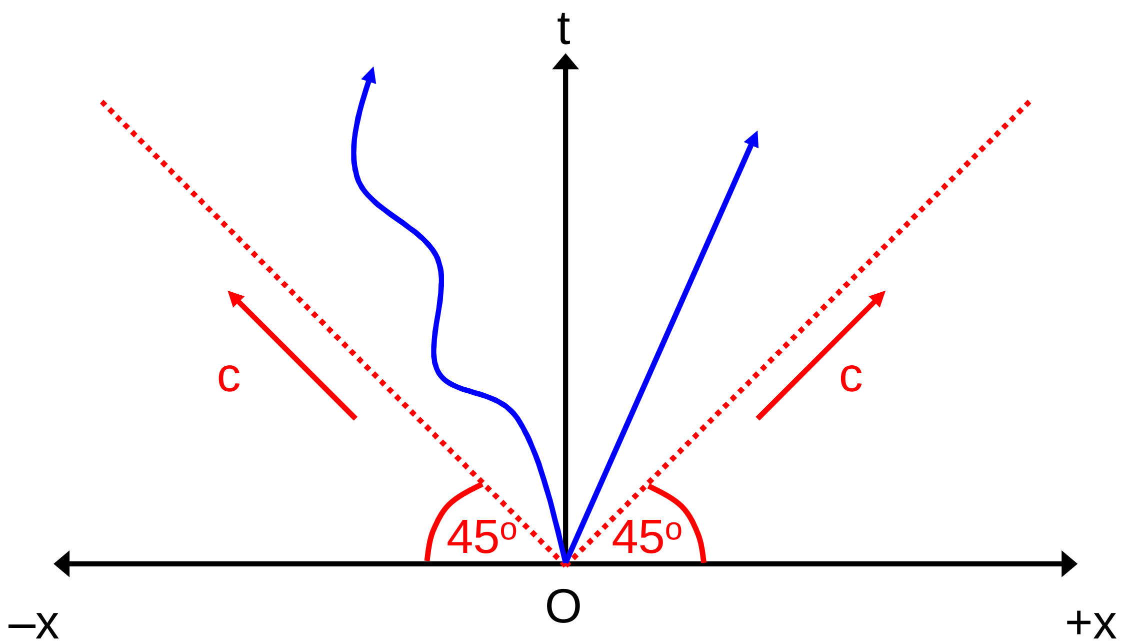
Plot of the Lorentz factor γ(v)



Notice the minimum value of 1 at a velocity (u in the graph) of zero. At zero velocity there is no relativistic effect at all. As speed increases (plotted here as increasing in either direction from the origin) then gamma rapidly increases, in fact it increases asymptotically, tending to +infinity as speed tends to the speed of light (set equal to one in this graph, simply by changing units of measurement). Gamma becomes infinite at the speed of light!

Spacetime Graphs and World-Lines

We can plot the motion of a particle on an ordinary spacetime graph. To keep things simple let us stick to considering just one spatial coordinate, x (ignoring y and z for the moment), then a spacetime diagram is a plot of x versus time, t:

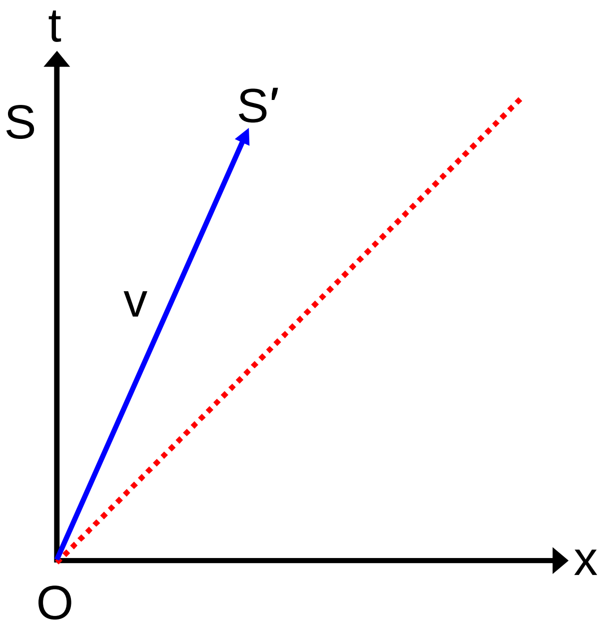


I have put x on the horizontal axis and time on the vertical. We set an arbitrary origin (O = (0,0)) for our coordinate system at which we set t= 0 and x = 0. This could, for example, be a certain position on a lab bench and time zero could be when we set an object into motion. I have plotted two paths taken through space and time by two objects (in blue) both of which began at the origin. One object has moved away at constant speed and moved to the right (+x), the other has taken a more convoluted path, accelerating in various ways, as it winds off to the left. These paths are the **world-lines** of the objects – the trajectories that they trace out in space and time.

Also shown are the paths of two light beams as the two red dotted lines, travelling away from the origin in either direction at constant speed c. Setting up the axis scales according to convention, these light beams are at 45 degrees to the x and t axes. If I added in another axis, say the y axis and draw it going into and out of the page, then our allowed light paths would form a light cone. We shall see later that Special Relativity forbids any normal object (any signal) from traveling faster than light, so no world-line can have an angle less than 45 degrees in our diagram. If we were stationary, then our world-line would simply be a straight-line along the t-axis – our position would not change!

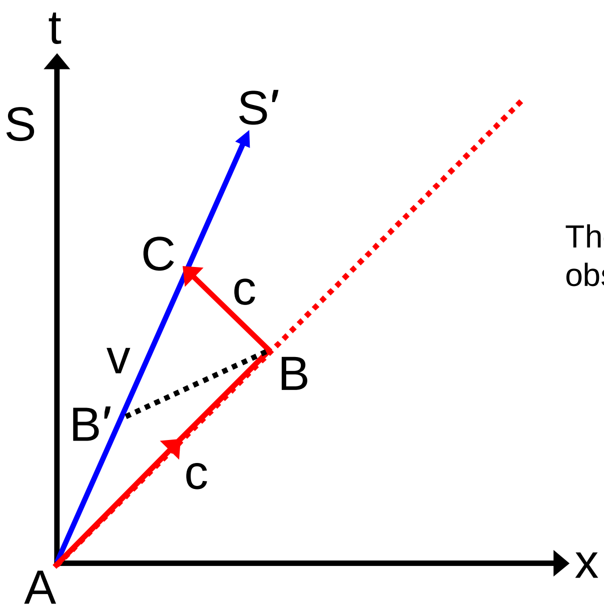
The Simultaneity Shift

Consider one stationary observer, S. their world-line is the t-axis. Consider another observer moving away from S at a constant velocity, v, call this moving observer S′ (S-prime). The world-line of S′ is a slanted line on our graph, which is really the graph that S sees:



To S′ it is S that is moving away (to all intents and purposes) and their graph looks identical but with the S and S′ labels switched. They have passed each other in outer space and who can be sure what their actual velocities are? Velocity is only relative!

Now suppose observer S′ shines a light beam at point A (which is at the origin) straight off to their side and toward a mirror at point B and then they intercept the reflected beam at C (both light-beams travel at speed c and at 45 degrees to the axes and are shown in red):



The line BB′ is a simultaneity line for observer S′.

When does the light signal arrive at B?

• Both S and S′ assume they are at rest.

• Looking at the graph, which is plotted from the perspective of S, the light takes longer to travel from A to B than it does to ravel from B to C. To S the light reaches B at time B.

• To S′ the light takes an equal time – they think they are rest and the distance to the mirror is fixed from their perspective, so AB = BC according to S′. S′ sees the light reach the mirror at time B′ exactly half-way along her world-line from A to C.

<u>Calculation of the slope of line BB′</u>

The slope of a line is the vertical distance divided by the horizontal distance.

Slope of $BB' = \dfrac{\Delta t}{\Delta x}$

The inclination (1/slope) of line $BB' = \dfrac{\Delta x}{\Delta t}$

The inclinations of the other lines are as follows:

(1) AB: $\dfrac{\boldsymbol{x}(\boldsymbol{B})}{\boldsymbol{t}(\boldsymbol{B})} = \boldsymbol{c}$

(2) AC: $\dfrac{\boldsymbol{x}(\boldsymbol{C})}{\boldsymbol{t}(\boldsymbol{C})} = \boldsymbol{v}$

(3) BC: $\dfrac{\boldsymbol{x}(\boldsymbol{C}) - \boldsymbol{x}(\boldsymbol{B})}{\boldsymbol{t}(\boldsymbol{C}) - \boldsymbol{t}(\boldsymbol{B})} = -\boldsymbol{c}$

(this is essentially, speed = distance/time).

This gives:

(4) $\boldsymbol{x}(\boldsymbol{B}) = \boldsymbol{ct}(\boldsymbol{B})$

From the graph:

(5) $\boldsymbol{x}(\boldsymbol{C}) = 2\boldsymbol{x}(\boldsymbol{B}')$

or

$\boldsymbol{x}(\boldsymbol{B}') = \dfrac{\boldsymbol{x}(\boldsymbol{C})}{2}$

and

(6) $\boldsymbol{t}(\boldsymbol{B}') = \dfrac{\boldsymbol{t}(\boldsymbol{C})}{2}$

Also from the graph :

$$(7)\ \frac{x(B)}{t(B)} = c$$

and

$$(8)\ \frac{x(C)}{t(C)} = v$$

From (3) :

$$x(C) - x(B) = -c(t(C) - t(B))$$

$$t(C) - t(B) = -\frac{x(C) - x(B)}{c}$$

$$t(C) = \frac{-x(C) - x(B)}{c} + t(B)$$

Substituting in (1) and (2) :

$$t(C) = -\frac{vt(C) - ct(B)}{c} + t(B)$$

$$t(C) = -\frac{v}{c}t(C) + 2t(B)$$

$$t(C)\left(1 + \frac{v}{c}\right) = 2t(B)$$

and so :

$$t(C) = \frac{2t(B)}{1 + \frac{v}{c}}$$

$$t(C) = \frac{2ct(B)}{c + v}$$

Since :

$$t(C) = 2t(B')$$

$$(9)\ \ t(B') = \frac{ct(B)}{c + v}$$

$$\frac{\Delta t}{\Delta x} = \frac{t(B) - t(B')}{x(B) - x(B')}$$

$$\text{using (9)}:$$

$$t(B) - t(B') = t(B) - \frac{ct(B)}{c+v} = \left(1 - \frac{c}{c+v}\right)t(B)$$

and

$$x(B) - x(B') = ct(B) - \frac{cvt(B)}{c+v} = \left(c - \frac{cv}{c+v}\right)t(B)$$

so

$$\frac{\Delta t}{\Delta x} = \frac{\left(1 - \frac{c}{c+v}\right)}{\left(c - \frac{cv}{c+v}\right)} = \frac{\frac{v}{c+v}}{\frac{c^2}{c+v}} = \frac{v}{c^2}$$

and

$$\Delta t = \frac{v\Delta x}{c^2}$$

which is the <u>simultaneity shift</u>.

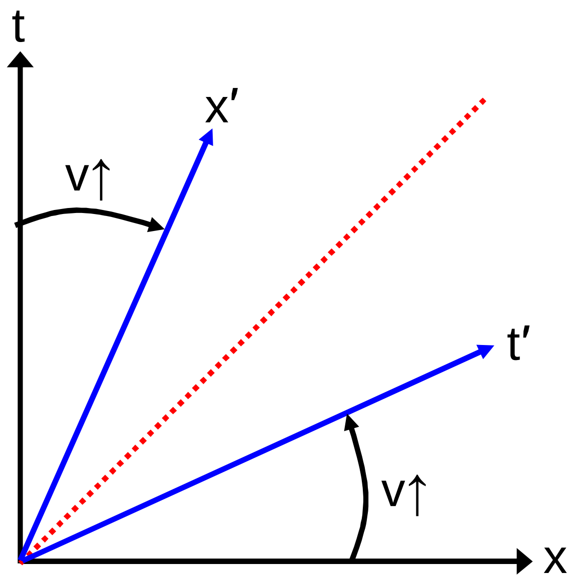
<u>What does the simultaneity shift mean</u>?

It means that the time-interval between two events appears different to different observers moving with different velocities.

Note:

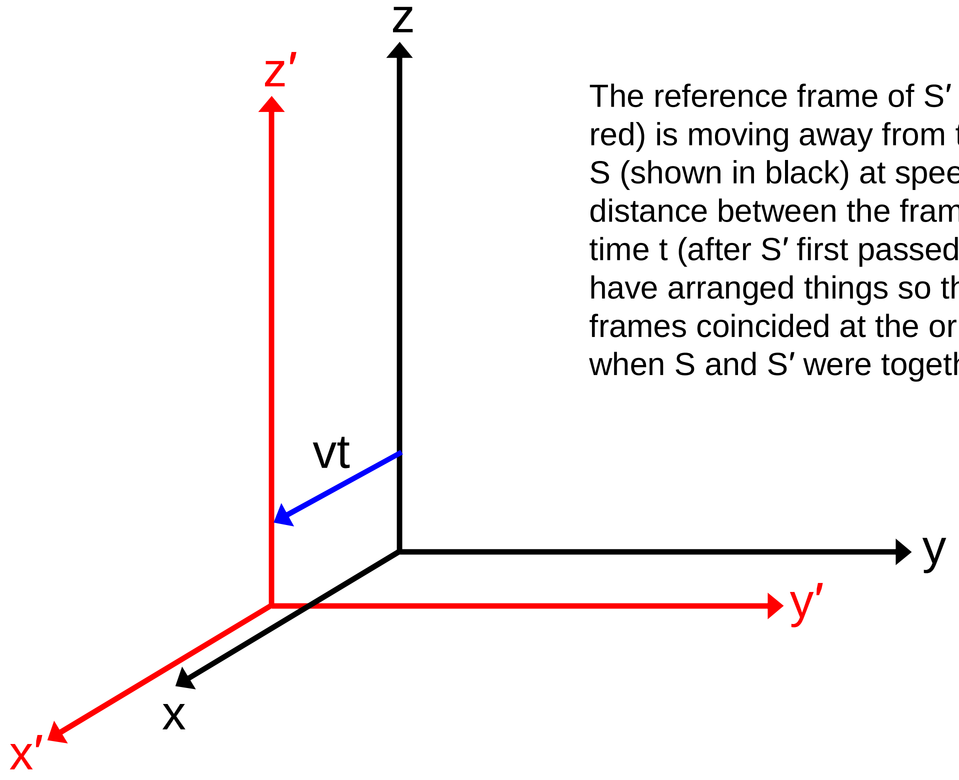
The line BB′ is the line along which events appear simultaneous to S′, it is parallel to their time axis. Notice that their x-axis, x′ runs along their world-line on their own version of the space-time graph, since they are stationary relative to themselves. Apparently simultaneous events appear to occur at the same time and so are on a line parallel to the time axis, t′ - both axes have become tilted for our traveler:

We shall represent space-time events in different ways. Plotting everything on the axes according to our ‘stationary’ observer, the axes of our moving observer appear compressed:



Above: the space-time axes of S′ are shown in blue, they are compressed as S′ is moving relative to S whose axes are at right-angles and shown in black. The higher the relative velocity, v, the more the axes compress, and at c they both coincide with the light-path (red dotted line).

We can also represent an event, such as observer S′ moving away from observer S along the x-axis at speed v:



The reference frame of S′ (shown in red) is moving away from the frame of S (shown in black) at speed v, so the distance between the frames at any time t (after S′ first passed S) is vt. We have arranged things so that both frames coincided at the origin at t = 0, when S and S′ were together.

Lorentz Transformation

We would like to be able take coordinates from one reference frame and map them to the other frame, that is take an event (t,x,y,z) in frame S and map it to the coordinates as observed by S′ and vice versa. This is a **Lorentz transformation**.

Such a transformation is not as simple as calculating Lorentz contractions and time dilations when we plot the moving reference frame on the axes of the 'stationary' frame as before, now we must take the relative movement of the frames into account.

We will transform the x and t coordinates, keeping y and z the same, implying relative motion along the x-axis only. Using all 3 spatial dimensions adds mathematical complexity without producing new physics, and we can always arrange axes so that the motion occurs along the x-axis (we are considering constant velocity motion, so motion at constant speed in a straight line).

$$\underline{\text{The Lorentz Transformation Equations:}}$$

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - (vx/c^2)}{\sqrt{1 - v^2/c^2}} = \gamma(t - (vx/c^2))$$

Here, the primed coordinates (t′,x′,y′,z′) are the coordinates of the event as seen by the observer, S′, we are considering to be moving (e.g. a person on a train) and (t,x,y,z) are the coordinates for the event as seen by the observer, S, whom we consider to be 'stationary' (e.g. someone waiting on the platform). Of course motion is relative, and we can not really say that S is stationary (they are on the earth which is moving!) but so long as we understand what we mean the physics is unchanged.

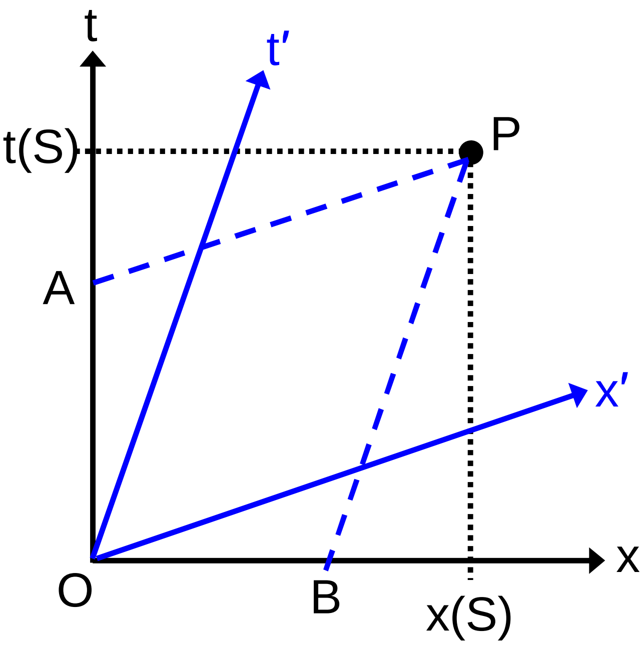
Where do these transformation equations come from?

Equation for x (refer to the graph above):

From the graph and the Lorentz contraction, S measures x according to the Lorentz contraction as:

$$\boldsymbol{x} = \boldsymbol{vt} + \boldsymbol{x'}\sqrt{1-\boldsymbol{v}2/\boldsymbol{c}2} = \boldsymbol{vt} + \frac{\boldsymbol{x'}}{\gamma}$$

Equation for t (refer to the space-time graph below):



Event P is recorded by S as occurring at (t(S),x(S)) = (x,t);

AP is the simultaneity line for S′ and is parallel to x′.

According to the simultaneity shift,

A occurs at time, t(A) :

$$\boldsymbol{t}(\boldsymbol{A}) = \boldsymbol{t} - \frac{\boldsymbol{vx}}{\boldsymbol{c}^2}$$

However, this is according to the time as read by S,

according to S′ this time, t′(A), is :

$$\boldsymbol{t'(A)} = \gamma t(A)$$

changing t′(A) for the equivalent symbol t′ :

$$t' = \gamma\left(1 - \frac{vx}{c^2}\right)$$

as required.

So now we have the correct equations for the transformation.

Galilean Transformation

Before the discovery of special relativity, physicists used the Galilean transformation to convert coordinates from a stationary frame to a moving one:

The Galilean Transformation Equations:

$$\boldsymbol{x'} = \boldsymbol{x} - \boldsymbol{vt}$$

$$\boldsymbol{y'} = \boldsymbol{y}$$

$$\boldsymbol{z'} = \boldsymbol{z}$$

$$\boldsymbol{t'} = \boldsymbol{t}$$

Notice that time was considered absolute and not relative!

If v is very small compared to c, γ is almost $= 1$,

(or if c is infinite, then $\gamma = 1$)

and the Lorentz transform becomes almost identical to the Galilean transform.

The Galilean transform is still a good approximation for typical daily speeds!

Causality and the Ultimate Speed Limit of the Universe

Below is a worked example of a Lorentz transform, with coordinates chosen to show a very important point about simultaneity and causality. (Don’t worry about the strange units given by Mathcad in the answers, they are odd but quite correct!).

Q. An observer in frame A observes two events, 1 and 2, to occur at the following coordinates:

Event 1: $x_1 := 10 \cdot 10^8 \cdot m$ $y_1 := 0$ $z_1 := 0$ $t_1 := 5 \cdot s$

Event 2: $x_2 := 30 \cdot 10^8 \cdot m$ $y_2 := 0$ $z_2 := 0$ $t_2 := 10 \cdot s$

Calculate the coordinates of the same events observed in frame B, if the speed of A relative to B is 80% the speed of light and the origins and axes of the frames coincided at time t = 0.

$c := 2.998 \cdot 10^8 \cdot \frac{m}{s}$ (speed of light)

$v := 0.8 \cdot c \rightarrow 239840000.0 \cdot \frac{m}{s}$

For event 1:

$$x'_1 := \frac{x_1 - v \cdot t_1}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow -332000000.0 \cdot m$$

$y'_1 := 0$ $z'_1 := 0$

$$t'_1 := \frac{t_1 - x_1 \cdot \frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow 3.885923949299533022 \cdot s$$

For event 2:

$$x'_2 := \frac{x_2 - v \cdot t_2}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow 1002666666.6666666667 \cdot m$$

$y'_2 := 0$ $z'_2 := 0$

$$t'_2 := \frac{t_2 - x_2 \cdot \frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow 3.3244385145652657327 \cdot s$$

In frame A, event 1 occurs before event 2.

In frame B event 2 occurs before event 1!

Spatial distance between events in frame A:

$$D_A := \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \rightarrow 2000000000 \cdot \sqrt{m^2}$$

Time interval between events in frame A

$$t_A := t_2 - t_1 \rightarrow 5 \cdot s$$

Light travel time between events in frame A:

$$L_A := \frac{D_A}{c} \rightarrow 6.671114076050700467 \cdot \frac{\sqrt{m^2}}{m} \cdot s$$

Spatial distance between events in frame B:

$$D_B := \sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2} \rightarrow 1334666666.6666666667 \cdot \sqrt{m^2}$$

Time interval between events in frame B

$$t_B := t'_2 - t'_1 \rightarrow -.5614854347342672893 \cdot s$$

Light travel time between events in frame B:

$$L_B := \frac{D_B}{c} \rightarrow 4.4518567934178341117 \cdot \frac{\sqrt{m^2}}{m} \cdot s$$

• Notice how in this example, the order of events observed by observer B was the opposite of the order of events observed by observer A!

• This implies that events 1 and 2 can not be causally linked! One event can not be the cause of the other, otherwise time has gone wrong, with the explosion occurring before the button is pressed according to frame B. This is because of the simultaneity shift. Events which occur simultaneously in one frame may not occur simultaneously in another, or events that occur in one order may occur in reverse order in another frame! What then is to stop observer B from sending a quick message to A telling them not to push the button because B just saw the explosion that would cause!

• If you look at the distance and time separating the events in each frame, and then look at how long it takes light to travel from one event to the other, then you will see that light is not fast enough to have traversed the space between the events in the time interval. This means that if no signal can travel faster than light then there is no way these two events can be causally linked, one can not be the cause of the other! They are quite separate events, so the order in which they are witnessed is immaterial. Note that B also could not relay a signal fast enough to A.

• Thus, **causality is preserved IF no signal can travel faster than light**.

• Since mass, and hence inertia, increase as an object speeds up, we can never accelerate an object to the speed of light simply by applying a force – it would take an infinite force! The object can reach very close to the speed of light but never quite reach it.

• Massless particles, however, do (almost always) travel at the speed of light, since they have no inertia.

If it is possible to travel faster than light, and more importantly, to send a signal faster than light, then according to Special Relativity all havoc can break loose! One observer of a horse race could observe it and then tell somebody else far away who the winner was and the message could reach the recipient before the race actually took place and then they would know in advance which horse was going to win!

This is why it is believed that no signal can travel faster than light. essentially, faster than light travel involves traveling to the past and that can cause a whole host of paradoxes!

Next we give a few other important results of Special relativity, though by no means all of them.

Relativistic Mass

In the sense that mass is a resistance to acceleration, or inertia, (it's harder to push a heavier object!) mass increases with speed! This is why an object with mass can not reach the speed of light (at least not by 'standard' means known to Earth physicists) because as it gets faster it gets harder and harder to push it faster!

We have a mss function, mass as a function of speed v, sometimes called the relativistic mass:

$$\boldsymbol{M}(\boldsymbol{v}) = \boldsymbol{m}\gamma$$

$$\boldsymbol{momentm}, \boldsymbol{p} = \boldsymbol{M}(\boldsymbol{v})\boldsymbol{v}$$

$$\text{and M(0) is the 'rest mass' of a particle.}$$

Proper Time

This is the time as given by the traveler's clock. Now, a traveler may not be moving at constant velocity, but at any instant they have a certain velocity relative to the Earth. What we do is consider an infinitesimal instant of time according to the traveler at this point, this is the proper time and is given by:

$$\boldsymbol{d}\tau = \frac{\boldsymbol{dt}}{\gamma}$$

Relativistic energy, rest energy and $E = mc^2$

For a non - relativistic object (moving at 'normal' speeds),
the kinetic energy, K is:

$$\boldsymbol{K} = \boldsymbol{mc}^2(\gamma - 1)$$

For low everyday speeds, that are not quite zero, we can approximate gamma:

$$\gamma \approx 1 + \frac{v^2}{2c^2}$$

and then:

$$\boldsymbol{K} = \frac{1}{2}\boldsymbol{mv}^2$$

which is the usual expression given, e.g. for the kninetic energy of a moving ball.

Note that for massless particles, like photons, K = pc, where p is the momentum of the photon. These particles have zero 'rest mass' and move at speed c. For these particles, $p = h/\lambda$, where h is Planck's constant and λ is the wavelength.

If we define the total energy of a particle as:

$$\boldsymbol{E} = \boldsymbol{mc}^2\gamma = \boldsymbol{M}(\boldsymbol{v})\boldsymbol{c}^2$$

then we can write:

$$\boldsymbol{E} = \boldsymbol{mc}^2 + \boldsymbol{mc}^2(\gamma - 1)$$

$$\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{K}$$

where we have defined:

$$\boldsymbol{E}_0 = \boldsymbol{mc}^2$$

as the 'rest mass' energy.

This famous equation gives us an equivalence between energy and matter, with one being completely convertible into the other.

•When your cells respire the fuels from the food you have eaten, they convert a very tiny fraction of the mass of those sugars etc. into energy via this equation!

•Nuclear power converts slightly more of the mass of the reacting atoms into energy.

•Matter/anti-matter annihilation can result in total conversion of mass to energy, releasing a tremendous amount of energy (due to the very high value of c^2), energy hidden and locked away in even a small amount of matter!

Evidence for Special Relativity

Special relativity is not just a theory, but a scientific theory, meaning that it makes predictions that have been tried and tested experimentally and there is a huge amount of experimental evidence supporting relativity. I shall mention just two here.

1. For example, highly accurate atomic clocks onboard commercial airlines slowed by some 39 nanoseconds during a flight from London to Washington and back! Clearly, at these relatively low speeds the effect is small, but nevertheless this effect was almost exactly the same as the predictions due to Special Relativity and General Relativity (accounting for Earth's gravity) and identical to within experimental error.

2. Another example is given y muon radiation. Energetic protons in the cosmic rays colliding with atoms in the Earth's atmosphere, sometime sproduce muons, a type of 'heavy electron' which is unstable, decaying into an electron, anti-electron neutrino and a mu-neutrino, with a half-life of 2.197×10^{-6} seconds.

$$\text{Muon decay :}$$
$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$$

Now, a detector on a mountain and a detector at sea level can both monitor the density of muons raining down upon the Earth from the upper atmosphere. Since muons decay very quickly, we would expect to see fewer at sea level than at mountain level, and we do. However, we don't see as few as we might naively expect, we see rather more, so fewer decay than expected!

The reason is that according to Special relativity, time is relative! The half-life we observe for a muon is not the same as that seen by muons raining down to Earth – these muons travel at about 98% the speed of light! Thus, by time dilation, a fast-moving muon will seem to last longer and travel further. Thus, they decay more slowly than we expect because they are moving fast and thus muon-time dilates and slows down! Lorentz calculations predict the observed decay-rate!

General Relativity

Special relativity assumes constant velocity, but what if one or more observers are accelerating? Special Relativity can be used to approximate these situations, since at any instance in time, the change in velocity will be small, and we can arrange a series of inertial reference frames along the path of the observer. However, forces, which cause acceleration, are really the subject of General Relativity which accounts for the force of gravity. We shall look at General Relativity in a future article.

<u>Traveling to the Stars Faster than light</u>?

Due to the problems of causality that we have highlighted, most physicists believe that it is impossible for any signal, including a massive object like a spaceship, to travel faster than light. However, Special Relativity does enable space travel over very great distances, by traveling forwards in time at a different rate, as in the Twin Paradox. The crew of a spaceship accelerating very close to the speed of light could travel to the edges of the known Cosmos in their life-times, in principle. The only trouble is, by the time they got there, billions of years would have elapsed! A space-fairing race would then be far from connected, but rather scattered in time and a starship would be unable to return home after any lengthy voyage!

Other theories have been and are being developed, however, which incorporate faster-than-light travel in ways which would not violate causality. String Theory predicts the existence of Tachyons, particles that travel faster than light. However, we perhaps could never detect tachyons and so never use them to send a message.

Some of these theories assume that the Lorentz symmetry, which gives us the transformations between different frames, is only approximate. Under certain conditions, perhaps at very high energies, there is some violation of this theory (which would also lead to CPT violation in particle physics). CPT symmetry (which is discussed in another article on Symmetries) states that a mirror-image of the Universe, in which all positions and momenta were reversed and in which matter would be replaced by anti-matter would follow identical laws. Certainly, no differences between particles and anti-particles (other than their electric charges) have been irrefutably observed; except that matter is much more abundant. However, there is some suggestion that antineutrinos may have slightly different masses than neutrinos, implying a violation. If this symmetry is not exact, but only approximate, then Lorentz invariance is also necessary violated. This would make relativity only an approximate theory.

In the **Superfluid Vacuum Theory** (SVT), the vacuum of space contains a sea of bosons that are condensed into a Bose-Einstein Condensate (BEC). This theory causes Lorentz symmetry to be only approximate, due to small vacuum-field fluctuations. This could allow massive particles to reach the speed of light at finite energy!

General relativity also opens doors to new possibilities. In General relativity the presence of energy causes space-time to bend or warp, accounting for gravity. The metric of a local region of space-time is a mathematical law defining the distance between two points in space and time (in spatial terms, calculating the distance between two points on a flat surface and two on the surface of a sphere require a different metric). The Schwarzschild metric, for example, describes the space-time curvature around a spherical (non-rotating) mass, such as a stationary star or black-hole. The **Alcubierre metric** describes a **warp bubble**, ahead of which spacetime contracts, and aft of which spacetime expands. Warping spacetime significantly requires massive amounts of energy, but perhaps some form of **warp drive** is possible.

Some theries also introduce additional spatial and temporal dimensions. String Theory may introduce 25 spatial dimensions and one of time, superstring theory has 10 or 11 dimensions, but still only one of time. Heim theory uses extra timelike dimensions of space and can apparently successfully predicts QED. The verdict is still out thee, perhaps waiting for the next Einstein!?

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